REAL ANALYSIS
TOPIC VIII - TOPOLOGICAL SPACES

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Abstract. We define *topological spaces*, which are sets together with the additional structure provided by a collection of open sets. We give examples and develop their basic properties.

Contents

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1. Topological Spaces

1.1. Topological Spaces.

Definition 1. A *topological space* is a set X together with a collection of subsets $\mathcal{T} \subset \mathcal{P}(X)$ such that

- **(T1)** $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;
- **(T2)** $\mathcal{U} \subset \mathcal{T} \Rightarrow \cup\mathcal{U} \in \mathcal{T}$;
- **(T3)** $\mathcal{U} \subset \mathcal{T}$ and $\mathcal{U}$ finite $\Rightarrow \cap\mathcal{U} \in \mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology* on X. A subset $A \subset X$ is called *open* if $A \in \mathcal{T}$, and is called *closed* if $X \smallsetminus A \in \mathcal{T}$.

Remark 1. To rephrase **(T2)** and **(T3)**, the union of any number of open sets is open, and the intersection of finitely many open sets is open. By DeMorgan's Laws, the intersection of any number of closed sets is closed, and the union of finitely many closed sets is closed.

Example 1. Let (X, d) be a metric space. Let $a \in X$ and $r > 0$. Recall that the *ball of radius r around a* is

$$B_r(a) = \{x \in X \mid d(x, a) < r\}.$$

A subset $U \subset X$ is called *open* if, for every $u \in U$, there exists $r > 0$ such that $B_r(u) \subset U$.

Let $\mathcal{T}$ denote the set of all subsets of X which are open. Then $\mathcal{T}$ is a topology on X, called the *metric topology*, and $(X, \mathcal{T})$ is a topological space. In this way, topological spaces generalize metric spaces.

Example 2. Let X be a set and let $\mathcal{T} = \mathcal{P}(X)$. Then $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ is called the *discrete* topology on X.

The discrete topology may be obtained from a metric. For example, if we set $d(x, x) = 0$ and $d(x, y) = 1$ if $x \neq y$, then metric topology is the discrete topology on X.

Example 3. Let X be a set and let $\mathcal{T} = \{\varnothing, X\}$. Then $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ is called the *trivial* topology on X.

Example 4. Let X be a set and let $\mathcal{T} = \{A \subset X \mid X \smallsetminus A \text{ is finite }\}$. Then $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ is called the *cofinite* topology on X.

Example 5. Let X be a set. A *tower* of subsets of X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ which contains the empty set and the entire set and is totally ordered by inclusion.

Let X be a set and $\mathcal{T}$ a tower of subsets of X. Then $\mathcal{T}$ is a topology on X, called a *tower topology*.

Example 6. Let X be a totally ordered set. For $a \in X$, set

$$L_a = \{x \in X \mid x < a\} \quad \text{and} \quad R_a = \{x \in X \mid x > a\}.$$

Set

$$\mathcal{L} = \{L_a \mid a \in X\} \cup \{\varnothing, X\} \quad \text{and} \quad \mathcal{R} = \{R_a \mid a \in X\} \cup \{\varnothing, X\}.$$

Then $\mathcal{L}$ is a topology on X, called the *left order topology*, and $\mathcal{R}$ is a topology on X, called the *right order topology*.

1.2. **Neighborhoods.**

Definition 2. Let X be a topological space and let $x \in X$. A *neighborhood* of x is a subset $N \subset X$ such that there exists an open set $U \subset N$ with $x \in U$.

Remark 2. Let X be a topological space and let $x \in X$. If U is an open set containing x, then U is itself a neighborhood of x, and is referred to as an open neighborhood. Thus there exists at least one neighborhood of x; indeed, X is open and contains x.

Definition 3. A *deleted neighborhood* of x is a set of the form $N \setminus \{x\}$, where N is a neighborhood of x.

Remark 3. Let X be a topological space and let $x \in X$. If $N \setminus \{x\}$ is a deleted neighborhood of x which does not intersect A, then either N does not intersect A or there is an open set $U \subset N$ such that x is the only element of A in that open set.

1.2.1. *Closure Points.*

Definition 4. Let X be a topological space and let $A \subset X$. A *closure point* of A is a point $x \in X$ such that every neighborhood of x intersects A. The *closure* of $A \subset X$ is the set of closure points of A and is denoted $\overline{A}$.

Remark 4. Clearly every neighborhood of x intersects A if and only if every open neighborhood of x intersects A.

Proposition 1. *Let X be a topological space. Let $A, B \subset X$. Then*

- **(a)** $\overline{\varnothing} = \varnothing$;
- **(b)** $A \subset \overline{A}$;
- **(c)** $\overline{\overline{A}} = \overline{A}$;
- **(d)** $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$.

Proof. If $x \in X$, then X is a neighborhood of x which does not intersect $\varnothing$; thus, $\varnothing$ has no points of closure, so $\overline{\varnothing} = \varnothing$.

If $a \in A$, then a is in the intersection of any neighborhood of a with A; thus $A \subset \overline{A}$.

From **(b)** we have $\overline{A} \subset \overline{\overline{A}}$. Suppose that $x \in \overline{\overline{A}}$. Then every open neighborhood of x intersects $\overline{A}$. For any open neighborhood U of x, let $y \in U \cap \overline{A}$. Then every open neighborhood of y intersects A. Since U is an open neighborhood of y, U intersects A. Thus $x \in \overline{A}$.

Suppose that $x \notin \overline{A} \cup \overline{B}$. Then there exists a neighborhoods U, V of x such that $U \cap A = \varnothing$ and $V \cap B = \varnothing$. Then $U \cap V$ is a neighborhood of x such that $(U \cap V) \cap (A \cup B) = \varnothing$. So $x \notin \overline{(A \cup B)}$. Therefore $\overline{(A \cup B)} \subset \overline{A} \cup \overline{B}$.

Suppose that $x \in \overline{A} \cup \overline{B}$. Then every open neighborhood of x intersects A or B, so it intersects $A \cup B$. Thus $x \in \overline{A} \cup \overline{B}$, so $\overline{A} \cup \overline{B} \subset \overline{(A \cup B)}$. □

Proposition 2. *Let X be a topological space. If $A \subset B \subset X$, then $\overline{A} \subset \overline{B}$.*

Proof. Let $y \in \overline{A}$. Then every neighborhood of y intersects A. Since $A \subset B$, every neighborhood of y intersects B. Thus $y \in \overline{B}$. □

Proposition 3. *Let X be a topological space and $A \subset X$. Then $\overline{A}$ is the intersection of the closed sets of X which contain A.*

Proof. Let $\mathcal{F}$ denote the set of all closed sets which contain A. We wish to show that $\overline{A} = \cap \mathcal{F}$.

We select $x \in X$. If $x \notin \cap \mathcal{F}$, then $x \notin F$ for some closed set F which contains A. If $U = X \smallsetminus F$, then U is open, so U is a neighborhood of x which does not intersect A. Thus x is not a closure point of A; that is, $x \notin \overline{A}$.

On the other hand, if $x \notin \overline{A}$, there is an open neighborhood U of x which does not intersect A. If $F = X \smallsetminus U$, then $x \notin F$, but F is a closed set which contains A, so $F \in \mathcal{F}$. Thus $x \notin \cap \mathcal{F}$. □

Proposition 4. *Let X be a topological space and $A \subset X$. Then A is closed if and only if $A = \overline{A}$.*

Proof. Suppose that A is closed. Then $U = X \smallsetminus A$ is open. If $x \notin A$, then U is a neighborhood of x which does not intersect A, so x is not a closure point of A; that is, $x \notin \overline{A}$. This shows that $\overline{A} \subset A$, and since we already know that $A \subset \overline{A}$, we conclude that $A = \overline{A}$.

Conversely, suppose that $A = \overline{A}$; we wish to show that the complement of A is open. Thus let $x \in X \smallsetminus A$. Then $x \notin \overline{A}$, so there is a neighborhood U of x which does not intersect A. So, x is interior to $X \smallsetminus A$, which shows that $X \smallsetminus A$ is open, so A is closed. □

1.2.2. *Interior Points.*

Definition 5. Let X be a topological space and let $A \subset X$. An *interior point* of A is a point $x \in X$ such that A contains a neighborhood of x. The *interior* of A is the set of interior points of A and is denoted A°.

Proposition 5. *Let X be a topological space and let $A \subset X$. Then A° is the union of the open sets contained in A.*

Proposition 6. *Let X be a topological space and let $A \subset X$. Then A is open if and only if $A = A^\circ$.*

Proposition 7. *Let X be a topological space and let $A \subset X$. Then*

- **(a)** $A^\circ = X \smallsetminus \overline{(X \smallsetminus A)}$;
- **(b)** $\overline{A} = X \smallsetminus (X \smallsetminus A)^\circ$;
- **(c)** $A \subset B \Rightarrow A^\circ \subset B^\circ$;
- **(d)** $(A^\circ)^\circ = A^\circ$;
- **(e)** $(A \cap B)^\circ = A^\circ \cap B^\circ$.

1.2.3. *Boundary Points.*

Definition 6. Let X be a topological space and let $A \subset X$. A *boundary point* of A is a point $x \in X$ such that every neighborhood of x intersects A and $X \setminus A$. The *boundary* of A is the set of boundary points of A and is denoted ∂A.

Proposition 8. *Let X be a topological space and let $A \subset X$. Then*

(a) $\partial A = \overline{A} \setminus A^\circ$;
(b) $\partial A = \overline{A} \cap \overline{(X \setminus A)}$;
(c) $\partial A = \partial(X \setminus A)$;
(d) $\overline{A} = A \cap \partial A$;
(e) $A^\circ = A \setminus \partial A$;
(f) $\partial(\partial A) \subset \partial A$;
(g) $A \cap B \cap \partial(A \cap B) = A \cap B \cap (\partial A \cup \partial B)$.

Proposition 9. *Let X be a topological space and let $A \subset X$. Then $\partial A = \varnothing$ if and only if A is both open and closed.*

Proof.
($\Rightarrow$) Suppose that $\partial A = \varnothing$. Then $\overline{A} \subset A^\circ$. But $A^\circ \subset A \subset \overline{A}$, so $A^\circ = A = \overline{A}$. Thus A is both open and closed.
($\Leftarrow$) Suppose that A is both open and closed. Then $A^\circ = A = \overline{A}$, so $\partial A = \overline{A} \setminus A^\circ = \varnothing$. □

1.2.4. *Accumulation Points.*

Definition 7. Let X be a topological space and let $A \subset X$. An *accumulation point* of A is a point $x \in X$ such that every deleted neighborhood of x intersects A. The *derived set* of A is the set of accumulation points of A, and is denoted A'.

Proposition 10. *Let X be a topological space and $A, B \subset X$.*

(a) $A \subset B \Rightarrow A' \subset B'$;
(b) $(A \cup B)' = A' \cup B'$;
(c) $\overline{A} = A \cup A'$.

Corollary 1. *A subset of a topological space is closed if and only if it contains all of its accumulation points.*

1.2.5. *Isolated Points.*

Definition 8. Let X be a topological space and let $A \subset X$. An *isolated point* of A is a point $x \in A$ such that some deleted neighborhood of x is contained in $X \setminus A$. The set of isolated points of A is denoted $A^\odot$.

Proposition 11. *Let X be a topological space and $A \subset X$.*

(a) $A^\odot \subset A$;
(b) $A^\odot \subset \partial A$;
(c) $\overline{A} = A' \sqcup A^\odot$.

1.3. **Refinements.**

1.3.1. *Refinements.*

Definition 9. Let X be a set and let $\mathcal{S}$ and $\mathcal{T}$ be topologies on X.

If $\mathcal{S} \subset \mathcal{T}$, we say that $\mathcal{S}$ is a *courser* topology than $\mathcal{T}$ and that $\mathcal{T}$ is a *finer* topology than $\mathcal{S}$.

Remark 5. The coarsest topology on a set is the trivial topology and the finest topology on a set is the discrete topology.

Proposition 12. *Let X be a set and let $\{\mathcal{T}_\alpha \mid \alpha \in A\}$ be a collection of topologies on X. Then $\mathcal{I} = \cap_{\alpha \in A} \mathcal{T}_\alpha$ is a topology on X.*

Proof. Since the empty set and the entire set are in every topology, they are in the intersection.

If $\mathcal{U} \subset \mathcal{I}$, then $\mathcal{U} \subset \mathcal{T}_\alpha$ for every α. Thus $\cup\mathcal{U} \in \mathcal{T}_\alpha$ for every α, so $\cup\mathcal{U} \in \mathcal{I}$.

If $\mathcal{U} \subset \mathcal{I}$, then $\mathcal{U} \subset \mathcal{T}_\alpha$ for every α. If $\mathcal{U}$ is a finite collection, $\cap\mathcal{U} \in \mathcal{T}_\alpha$ for every α, so $\cap\mathcal{U} \in \mathcal{I}$. □

1.3.2. *Generated Topologies.*

Definition 10. Let X be a set and let $\mathcal{A} \in \mathcal{P}(X)$.

The *topology generated by* $\mathcal{A}$ is the intersection of all the topologies on X which contain $\mathcal{A}$, and is denoted $\langle \mathcal{A} \rangle$.

Remark 6. The topology generated by a collection $\mathcal{A} \subset \mathcal{P}(X)$ is the coarsest topology on X in which all of the sets in $\mathcal{A}$ are open.

1.3.3. *Bases.*

Definition 11. Let X be a set.

A *basis* for a topology on X is a collection of subsets $\mathcal{B} \subset \mathcal{P}(X)$ such that

- **(B1)** $\cup\mathcal{B} = X$;
- **(B2)** if $B_1, B_2 \in \mathcal{B}$ and $p \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ with $p \in B_3$ and $B_3 \subset B_1 \cap B_2$.

Proposition 13. *Let X be a set and let $\mathcal{B}$ be a basis for a topology on X. Let $\mathcal{T}$ be the collection of unions of sets in $\mathcal{B}$. Then $\mathcal{T}$ is a topology on X, and $\mathcal{T} = \langle \mathcal{B} \rangle$.*

Proof. We consider the empty set to be the empty union, so $\varnothing \in \mathcal{T}$. By basis property **(B1)**, the entire set is in $\mathcal{T}$. By definition of $\mathcal{T}$, any union of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

Let $U_1, U_2 \in \mathcal{T}$. Since U_1 and U_2 are the unions of basis sets, for every $p \in U_1 \cap U_2$ there exist basis sets $B_1 \subset U_1$ and $B_2 \subset U_2$ such that $p \in B_1 \cap B_2$. By basis property **(2)**, there exists a basis set $B_p \subset B_1 \cap B_2 \subset U_1 \cap U_2$ with $p \in B_p$. Then

$$U_1 \cap U_2 = \bigcap_{p \in U_1 \cap U_2} \{p\} \subset \bigcap_{p \in U_1 \cap U_2} B_p \subset U_1 \cap U_2,$$

so $U_1 \cap U_2 = \bigcap_{p \in U_1 \cap U_2} B_p$ is open. □

Remark 7. Every topology on a set X has itself as a basis. Bases are not necessarily unique.

1.3.4. *Subbases.*

Definition 12. Let X be a set.

A *subbasis* for a topology on X is a collection of subsets $\mathcal{S} \subset \mathcal{P}(X)$ such that the collection of all finite intersections of sets in $\mathcal{S}$ form a basis for a topology on X.

Remark 8. Every basis is a subbasis.

Proposition 14. *Let X be a set and let $\mathcal{S} \subset \mathcal{P}(X)$. If $\cup\mathcal{S} = X$, then $\mathcal{S}$ is a subbasis for a topology on X. Let $\mathcal{B}$ be a basis which is the collection of all finite intersections of sets in $\mathcal{S}$. Then $\langle\mathcal{S}\rangle = \langle\mathcal{B}\rangle$.*

Proof. Basis property **(B1)** is given and basis property **(B2)** is obvious. The second claim follows immediately from the first. □

Remark 9. If $\mathcal{T}$ is a topology on X, then $\mathcal{S} \subset \mathcal{P}(X)$ is a subbasis for $\mathcal{T}$ if and only if $\langle\mathcal{S}\rangle = \mathcal{T}$.

1.3.5. *Equivalent Bases.*

Definition 13. Let X be a set.

Two subbases $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{P}(X)$ are *equivalent* if they generate the same topology on X.

Proposition 15. *Let X be a set with bases $\mathcal{B}_1$ and $\mathcal{B}_2$. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent if and only if for $p_1 \in B_1 \in \mathcal{B}_1$ there exists $B_2 \in \mathcal{B}_2$ such that $p_1 \in B_2 \subset B_1$ and for $p_2 \in B_2 \in \mathcal{B}_2$ there exists $B_1 \in \mathcal{B}_1$ such that $p_2 \in B_1 \subset B_2$.*

Proof. Let $\mathcal{T}_1 = \langle\mathcal{B}_1\rangle$ and $\mathcal{T}_2 = \langle\mathcal{B}_2\rangle$. Let $U \in \mathcal{T}_1$. Then $U = \cup_{\alpha\in A} B_\alpha$ for some $\{B_{1,\alpha} \mid \alpha \in A\} \subset \mathcal{B}_1$. For each $p \in U$ there exists $B_{1,\alpha}$ such that $p \in B_{1,\alpha}$. By hypothesis there exists $B_{2,\alpha} \in \mathcal{B}_2$ such that $p \in B_{2,\alpha} \subset B_{1,\alpha}$. Thus U is the union of such $B_{2,\alpha}$ and in $\mathcal{T}_2$. Therefore $\mathcal{T}_1 \subset \mathcal{T}_2$. Similarly, $\mathcal{T}_2 \subset \mathcal{T}_1$. □

Proposition 16. *Let X be a totally ordered set. For $a \in X$, let $L_a = \{x \in X \mid x < a\}$. and let $R_a = \{x \in X \mid x > a\}$. Then $\mathcal{S} = \{L_a \mid a \in X\} \cup \{R_a \mid a \in X\}$ is a subbasis for a topology on X.*

For $a, b \in X$, let $I_{a,b} = \{x \in X \mid a < x < b\}$. Let $\mathcal{B} = \{I_{a,b} \mid a, b \in X\}$. Then $\mathcal{B}$ forms the basis for a topology on X which is generated by the subbasis $\mathcal{S}$. The topology generated by $\mathcal{B}$ is called the total order topology *on X.*

Remark 10. The standard topology on the real numbers is a total order topology.

2. Continuous Functions

2.1. **Continuous Functions.**

Definition 14. Let X and Y be spaces and $f : X \to Y$.

We say that f is *continuous* if for every open set $V \subset Y$, $f^{-1}(V) \subset X$ is open.

Definition 15. Let X and Y be spaces and $f : X \to Y$ and let $x_0 \in X$.

We say that f is *continuous at x_0* if for every neighborhood V of $f(x_0)$ there exists a neighborhood U of x_0 such that $f(U) \subset V$.

Proposition 17. *Let X and Y be spaces and $f : X \to Y$. Then f is continuous if and only if f is continuous at every point in X.*

Proof. Suppose that f is continuous, and let $x_0 \in X$. Let V be a neighborhood of $y_0 = f(x_0)$. Then $U = f^{-1}(V)$ is a neighborhood of x_0 which maps into V.

Conversely, suppose that f is continuous at every point in X. Let $V \subset Y$ be open and let $U = f^{-1}(V)$. For every $x \in U$, V is a neighborhood of $f(x)$, so there exists an open neighborhood U_x of x such that $f(U_x) \subset V$. But then $U_x \subset U$, and U is the union of such sets; thus U is open, and f is continuous. □

Proposition 18. *Let X and Y be spaces and $f : X \to Y$. If X has the discrete topology or Y has the trivial topology then f is continuous.*

Proposition 19. *Let X and Y be spaces and $f : X \to Y$. Then f is continuous if and only if for every closed set $F \subset Y$, $f^{-1}(F) \subset X$ is closed.*

Proof.

($\Rightarrow$) Suppose that f is continuous. Let $F \subset Y$ be closed. Let $U = Y \smallsetminus F$; then U is open, so $f^{-1}(U)$ is open, so $X \smallsetminus f^{-1}(U)$ is closed. But $X \smallsetminus f^{-1}(U) = f^{-1}(Y \smallsetminus U) = f^{-1}(F)$.

($\Leftarrow$) Suppose that for every closed set $F \subset Y$, $f^{-1}(F)$ is closed in X. Let $U \subset Y$ be open; then $Y \smallsetminus U$ is closed in Y, so $f^{-1}(Y \smallsetminus U)$ is closed in X. Thus $f^{-1}(U) = X \smallsetminus f^{-1}(Y \smallsetminus U)$ is open in X. Therefore f is continuous. □

Proposition 20. *Let X and Y be spaces and $f : X \to Y$. Then f is continuous if and only if for every $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.*

Proof.

($\Rightarrow$) Suppose that f is continuous. Let $A \subset X$ and let $y \in f(\overline{A})$. Then $y = f(x)$ for some point $x \in \overline{}(A)$. Let V be an open neighborhood of y. Then $f^{-1}(V)$ is open in X and contains x. Thus there exists $a \in A \cap f^{-1}(V)$, and $f(a) \in V$; that is, V intersects $f(A)$. Therefore $y \in \overline{f(A)}$, and $f(\overline{A}) \subset \overline{f(A)}$.

($\Leftarrow$) Suppose that for every $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.

Let $F \subset Y$ be closed and let $A = f^{-1}(F)$. Then $f(A) = F$, and since F is closed, $\overline{f(A)} = F$. Thus $F = f(A) \subset f(\overline{A}) \subset \overline{f(A)} = F$. This shows that $f(\overline{A}) = F$, so $\overline{A} \subset f^{-1}(F) = A$; since $A \subset \overline{A}$, we see that $A = \overline{A}$, so A is closed. Therefore f is continuous. □

Proposition 21. *Let X and Y be topological spaces and let $f : X \to Y$ be continuous. Let $A \subset X$. Then*

(a) $f(A)^\circ \subset f(A^\circ)$;
(b) $f(A)^\odot \subset f(A^\odot)$.

Proof. Let $y \in f(A)^\circ$. Then $y \in f(A)$, so $y = f(x)$ for some $x \in X$. Also there exists an open set V in Y such that $y \in V \subset f(A)$. Since f is continuous, $f^{-1}(V) \subset A$ is an open neighborhood of x contained in A, so $x \in A^\circ$, and $y \in f(A^\circ)$. Therefore $f(A)^\circ \subset f(A^\circ)$; this proves **(a)**.

Let $y \in f(A)^\odot$). Then $y \in f(A)$, and there exists an open neighborhood V of y in Y such that $V \cap (f(A) \smallsetminus \{y\}) = \varnothing$. Then $f^{-1}(V) \cap A = \{y\}$, so $y \in f(A)^\odot$. Therefore $f(A)^\odot \subset f(A^\odot)$; this proves **(b)**. □

2.2. **Open and Closed Maps.**

Definition 16. Let X and Y be spaces and let $f : X \to Y$.

We say that f is *open* if for every open set $U \subset X$, $f(U) \subset Y$ is open.

We say that f is *closed* if for every closed set $F \subset X$, $f(F) \subset Y$ is closed.

Proposition 22. *Let X and Y be topological spaces and let $f : X \to Y$ be a function. Let $\mathcal{B}$ be a basis for X. If $f(B)$ is open in Y for every $B \in \mathcal{B}$, then f is an open map.*

Proof. Let $U \subset X$ be open. Then $U = \cup_{\alpha \in I} B_\alpha$ for some collection $\{B_\alpha \in \mathcal{B} \mid \alpha \in I\}$ of basis sets. Thus $f(U) = f(\cup_{\alpha \in I} B_\alpha) = \cup_{\alpha \in I} f(B_\alpha)$; since $f(B_\alpha)$ is open for every $\alpha \in I$, then so is $f(U)$. □

Example 7. Let $X = \{(x, y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\}$ and let $Y = \mathbb{R}$. Let $f : X \to Y$ by $f(x, y) = x$. Then f is a surjective continuous closed map which is not open. It is not open because, for example, the set $\{(0, y) \mid y \in (1, 2)\}$ is open in X but projects onto a point in $\mathbb{R}$.

Example 8. Let $X = \mathbb{R}$ and $Y = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Define $f : X \to Y$ by $f(x) = (\cos x, \sin x)$. Then f is a surjective continuous open map which is not closed. It is open because it is open on a basis for the topology of $\mathbb{R}$ consisting of open intervals whose width is less than 2π. It is not closed because, for example, the set $\{x \in \mathbb{R} \mid x = 2\pi n + \frac{\pi}{2n}\}$ is closed in X but its image in Y has an accumulation point $(1, 0)$ which is not in the image.

Definition 17. Let X and Y be spaces and let $f : X \to Y$. We say that f is *bicontinuous* if it is both open and continuous.

Proposition 23. *Let $f : X \to Y$ be a bijective function between spaces. Then f is open if and only if f^{-1} is continuous.*

2.3. **Homeomorphisms.**

Definition 18. Let X and Y be spaces. A *homeomorphism* between X and Y is a function $f : X \to Y$ which is bijective, open, and continuous. If there exists a homeomorphism between X and Y, we say that X and Y are *homeomorphic.*

Proposition 24. *Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be topological spaces. Then X and Y are homeomorphic if and only if there exists an inclusion preserving bijection between $\mathcal{S}$ and $\mathcal{T}$.*

Proposition 25. *The empty space is unique. Up to homeomorphism, a one point space is unique.*

Proposition 26. *Let* $S = \{0, 1\}$ *and* $\mathcal{T} = \{\varnothing, \{0\}, S\}$*. Then* $\mathcal{T}$ *is a topology on* S *and* $(S, \mathcal{T})$ *is called the* Sierpinski space*.*

Proposition 27. *Up to homeomorphism, there are exactly three spaces with two elements.*

Proof. Let $X = \{a, b\}$. The trivial space and the discrete space on X are clearly distinct.

The only other possibilities for a topology on X are $\mathcal{T}_a = \{\varnothing, \{a\}, X\}$ are $\mathcal{T}_b = \{\varnothing, \{b\}, X\}$. Permuting a and b is a homeomorphism between $(X, \mathcal{T}_a)$ and $(X, \mathcal{T}_b)$. □

Remark 11. In the above notation, sending a to 0 and b to 1 is a homeomorphism between $(X, \mathcal{T}_a)$ and the Sierpinski space. Homeomorphisms preserve all properties relevant to topology. For this reason, $(X, \mathcal{T}_a)$ and $(X, \mathcal{T}_b)$ may also be called the Sierpinski space.

3. Subspaces, Products and Quotients

3.1. **Subspaces.**

Definition 19. Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$.

The *subspace topology* relative to Y is collection of subsets of Y

$$\mathcal{T}(Y) = \{U \cap Y \mid U \in \mathcal{T}\}.$$

Proposition 28. *Let* $(X, \mathcal{T})$ *be a topological space and let* $Y \subset X$*.*
Then $(Y, \mathcal{T}(Y))$ *is a topological space, called a* subspace *of* $(X, \mathcal{T})$*. The open sets of* $(Y, \mathcal{T}(Y))$ *are called* relatively open *in* X *and the closed sets are called* relatively closed *in* X*.*

Remark 12. Every subspace of a trivial, discrete, cofinite, or cocountable space is respectively trivial, discrete, cofinite, or cocountable.

Proposition 29. *Let* $(X, \mathcal{T})$ *be a topological space and let* $Y \subset X$*. Then* $\mathcal{T}(Y) \subset \mathcal{T}$ *if and only if* $Y \in \mathcal{T}$*.*

Definition 20. Let X be a space and let Y be a subspace of X. Then the map $i : Y \to X$ defined by $y \mapsto y$ is called the *inclusion map*.

Proposition 30. *Let* X *be a space and let* Y *be a subspace of* X*. Then the inclusion map* $i : Y \to X$ *is continuous.*

Definition 21. Let X and Y be spaces and let $f : X \to Y$.

We say that f is *relatively open* if for every open $U \subset X$, $f(U)$ is relatively open in $f(X)$.

We say that f is an *embedding* if f is injective, continuous, and relatively open.

Proposition 31. *Let* $f : X \to Y$ *be an embedding. Then* X *is homeomorphic to the subspace* $f(X) \subset Y$*.*

3.2. **Products.**

Definition 22. Let $\mathcal{X}$ be a nonempty family of topological spaces and let $\times\mathcal{X}$ be their Cartesian product. For $X \in \mathcal{X}$, let π_X be the projection from $\times\mathcal{X}$ onto X.

Let $\mathcal{S} = \{\pi_X^{-1}(U) \mid X \in \mathcal{X} \text{ and } U \text{ open in } X\}$. Then $\langle\mathcal{S}\rangle$ is called the *product topology* on $\times\mathcal{X}$.

A cartesian product of a family of spaces endowed with the product topology is called the *product space* of the family.

Proposition 32. *The product of a family of trivial spaces is trivial. The product of a family of discrete spaces is discrete if and only if the family is finite.*

Proposition 33. *Let $\mathcal{X} = \{X_\alpha \mid \alpha \in A\}$ be a nonempty family of spaces and let $\mathcal{B}$ be the set of cartesian products of one open set from each space such that all but finitely many of these open sets are the entire space. Then $\mathcal{B}$ is a basis for the product topology on $\times\mathcal{X}$.*

Proposition 34. *Let $\mathcal{X}$ be a family of spaces and let $\times\mathcal{X}$ be endowed with the product topology. Then for every $X \in \mathcal{X}$, the projection function $\pi_X : \times\mathcal{X} \to X$ is continuous.*

Proposition 35. *Let $\mathcal{X}$ be a family of spaces and let $\times\mathcal{X}$ be the cartesian product endowed with some topology. Suppose that for every $X \in \mathcal{X}$, the projection function $\pi_X : \times\mathcal{X} \to X$ is continuous. Then the topology on $\times\mathcal{X}$ is at least as fine as the product topology.*

3.3. **Quotients.**

Definition 23. Let $(X, \mathcal{T})$ be a topological space, Y a set, and $q : X \to Y$ a surjective function. Let $\mathcal{V} = \{V \subset Y \mid f^{-1}(V) \in \mathcal{T}\}$. Then $\mathcal{V}$ is a topology on Y, called the *quotient topology* on Y induced by q. The function q is called the or the *quotient projection* of X onto Y.

Proposition 36. *A quotient projection in continuous.*

Proposition 37. *The quotient topology is the finest topology such that the quotient projection is continuous.*

Proposition 38. *Let $\mathcal{X}$ be a family of topological spaces and let $\times\mathcal{X}$ be the product space. For $X \in \mathcal{X}$, let $\pi_X : \times\mathcal{X} \to X$ be the cartesian projection. Then X is endowed with the quotient topology induced by π_X.*

4. Connectedness

4.1. **Clopen Sets.**

Definition 24. Let X be a topological space and let $A \subset X$.

We say that A is *clopen* if A is both open and closed.

Proposition 39. *Let X be a topological space and let $A \subset X$. Then A is clopen if and only if for every $c \in \overline{A}$ there exists a neighborhood U of c such that $U \subset A$.*

4.2. **Separation.**

Definition 25. Let X be a space and let $A, B \subset X$.

We say that A and B are *separated* if there exist disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$. This relationship is denoted $A|B$. The pair (U, V) is called a *separation* of (A, B).

Proposition 40. *In a trivial space two sets are separated if and only if one of them is empty. In a discrete space two sets are separated if and only if they are disjoint. In a cofinite space two nonempty sets are separated if and only if they are disjoint and finite.*

Proposition 41. *Let X be a space and let $A, B \subset X$. Then $A|B$ if and only if $\overline{A} \cap B = A \cap \overline{B} = \varnothing$.*

Proof.

($\Rightarrow$) Let (U, V) be a separation of (A, B). Then for any $b \in B$, V is a neighborhood of b which is disjoint from A. Thus b in not in $\overline{A}$. Similarly, A does not intersect $\overline{B}$.

($\Leftarrow$) Suppose $\overline{A} \cap B = A \cap \overline{B} = \varnothing$. Let $U = X \setminus \overline{B}$ and $V = X \setminus \overline{A}$. Then U and V are open and form a separation of A and B. □

Proposition 42. *Let X be a space with subsets A, B, and C. Then*

- **(a)** $\varnothing|A$;
- **(b)** $A|B \Rightarrow B|A$;
- **(c)** $A|B$ *and* $C \subset A \Rightarrow C|B$;
- **(d)** $A|B$ *and* $A|C \Rightarrow A|(B \cup C)$;

Proposition 43. *Let X be a space and let $A, B \subset X$. The following conditions are equivalent:*

- **i.** $A|B$;
- **ii.** *A and B are disjoint relatively closed subsets of $A \cup B$;*
- **iii.** *A and B are disjoint relatively open subsets of $A \cup B$.*

Proof.

(i)$\Rightarrow$(ii) Suppose $\overline{A} \cap B = A \cap \overline{B} = \varnothing$. Since $A \subset \overline{A}$, A and B are disjoint. Since no point in B is in the closure of A, every point in B has a neighborhood which is disjoint from A. Let V be the union of these neighborhoods. Then V is open, $B \subset V$, and $V \cap A = \varnothing$. Similarly there exists and open set U such that $A \subset U$ and $U \cap B = \varnothing$. Then $U \cap A = A$ is relatively open in $A \cup B$ and so is $V \cap B = B$.

(ii)$\Rightarrow$(iii) Suppose that A and B are disjoint relatively closed sets in $A \cup B$. Then $A \cup B \setminus B = A$ and $A \cup B \setminus A = B$ are relatively open.

(iii)$\Rightarrow$(i) Suppose that A and B are disjoint relatively open sets in $A \cup B$. Then there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$, and $A \cap V = B \cap U = \varnothing$. Then no point of A is in the closure of B and vice versa. Thus $\overline{A} \cap B = A \cap \overline{B} = \varnothing$. □

4.3. **Connectedness.**

Definition 26. Let X be a space and let $A \subset X$.

We say that A is *separated* if it is the union of two nonempty separated sets, and A is *connected* if it is not separated.

Proposition 44. *Every subset of a trivial space is connected. A subset of a discrete space is connected if and only if it contains at most one element.*

Proposition 45. *Let X be a space. The following conditions are equivalent.*

- **i.** *X is connected;*
- **ii.** *X is not the disjoint union of two nonempty open sets;*
- **iii.** *X is not the disjoint union of two nonempty closed sets;*
- **iv.** *if $A \subset X$ is nonempty and clopen, then $A = X$.*

Proof. (i)⇒(ii) Suppose that $X = U \cup V$ with U, V open and disjoint. Then U and V are neighborhoods of the points they contain, and form a separation.

(ii)⇒(iii) Suppose that $X = C \cup F$ with C, F closed and disjoint. Let $U = X \setminus F$ and $V = X \setminus C$. Then U and V are open and disjoint and their union is X.

(iii)⇒(iv) Let C be a proper nonempty subset of X which is clopen. Then $F = X \setminus C$ is closed.

(iv)⇒(i) Suppose X is not connected. Then X is the union of nonempty separated sets A and B. Let (U, V) be a separation of (A, B). Then $A \subset U$, $B \cap U = \varnothing$, and $A \cup B = X$ together imply that $X = U \cup B$. Thus B is closed. Similarly, A is closed so B is also open. □

Proposition 46. *Let X be a space and let $A, B, C \subset X$. Suppose that $A|B$, C is connected, and $C \subset A \cup B$. Then $C \subset A$ or $C \subset B$.*

Proposition 47. *Let $\mathcal{Q}$ be a collection of connected subsets of a space X and C a connected subset of X which is not separated from any member of $\mathcal{Q}$. Then $C \cup (\cup \mathcal{Q})$ is connected.*

Proof. Let $Y = C \cup (\cup \mathcal{A})$ and suppose that $Y = A \cup B$ for some separated sets A and B. Thus $C \subset A$ or $C \subset B$. Suppose, without loss of generality, that $C \subset A$. Then for $Q \in \mathcal{Q}$, since Q is connected, $Q \subset A$ or $Q \subset B$. If $Q \subset B$, then $C|Q$, contrary to our hypothesis. Thus $Q \subset A$. Thus $A = Y$ and $B = \varnothing$. Thus Y is connected. □

Corollary 2. *Let $\mathcal{Q}$ be a collection of connected subsets of a space X. If $\cap \mathcal{Q}$ is nonempty, then $\cup \mathcal{Q}$ is connected.*

Proposition 48. *Let C be a connected subset of a space X and suppose that $C \subset A \subset \overline{C}$. Then A is connected. In particular, $\overline{C}$ is connected.*

Proof. Let U and V be disjoint open sets such that $A \subset U \cup V$. Then $C \subset U$ or $C \subset V$. Suppose, without loss of generality, that $C \subset U$. If $a \in A \cap V$, then V is a neighborhood of a which is disjoint from C and $a \notin \overline{C}$, contradicting our hypothesis. Thus $A \subset U$, and A cannot be separated. □

Proposition 49. *The continuous image of a connected set is connected.*

Proof. Let $f : X \to Y$ be a continuous function Let $C \subset X$ and let $D = f(C)$. Suppose that $D \subset f(X) \subset Y$ is not connected. Then there exist disjoint open sets $V_1, V_2 \subset Y$ such that $V_1 \cap D$ and $V_2 \cap D$ are nonempty. Then $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$ are disjoint open sets in X and $U_1 \cap C$ and $U_2 \cap C$ are nonempty. This implies that C is not connected. □

Proposition 50. *Let $\mathcal{X}$ be a family of spaces and let $\times\mathcal{X}$ be the Cartesian product endowed with the product topology. If every $X \in \mathcal{X}$ is connected, then $\times\mathcal{X}$ is connected.*

4.4. **Components.**

Proposition 51. *The connected subsets of a space X are partially ordered by inclusion.*

Definition 27. A *component* of a topological space is a maximal connected subset.

Proposition 52. *The only component of a connected space is the space itself. The components of a discrete space are the singleton sets.*

Proposition 53. *The components of a space partition the space.*

Proof. Let $x \in X$ and let C be the union of all connected subsets of X which contain x. Then C is connected and is clearly maximal. Thus X is the union of its components.

Let D be another component of X. If $C \cap D$ is nonempty, then $C \cup D$ is connected, and by the maximality of C, $D = C$. □

Proposition 54. *If C is a component of a space X, then C is closed.*

Proof. Since C is connected, so is $\overline{C}$. Since $C \subset \overline{C}$ and C is a maximal connected set, $C = \overline{C}$. □

Proposition 55. *Two distinct components of a space are separated.*

Proof. Let C and D be components of a space X and suppose that they are not separated. Then $\overline{C} \cap D$ or $C \cap \overline{D}$ is nonempty. Suppose, without loss of generality, that $\overline{C} \cap D$ is nonempty and let $x \in \overline{C} \cap D$. Since $C = \overline{C}$, $x \in C$. Thus $x \in C \cap D$ so $C = D$. □

Proposition 56. *If C is a component of a space X and $C \subset Y \subset X$, then C is a component of the subspace Y.*

4.5. **Dedekind Property.**

Definition 28. An ordered set X has the *Dedekind* property provided that for each decomposition $X = A \cup B$, where A and B are nonempty and $a < b$ whenever $a \in A$ and $b \in B$, either A contains a maximal element or B contains a minimal element, but not both.

Proposition 57. *A nonempty connected subset of an ordered space is infinite.*

Proposition 58. *An ordered space is connected if and only if it has the Dedekind property.*

Corollary 3. *Intervals of real numbers are connected.*

4.6. **Path Connectedness.**

Definition 29. Let X be a space. Let $I = [0, 1]$ be the unit interval in $\mathbb{R}$. A *path* in X is a continuous function $\gamma : I \to X$. The points $a = \gamma(0)$ and $b = \gamma(1)$ are called the *endpoints* of the path, and γ is referred to as a path between a and b.

Definition 30. A space X is *path connected* if for every two points in X there is a path between them.

Proposition 59. *If X is path connected, then X is connected.*

Proof. Suppose X is not connected. Then there exist disjoint nonempty open sets $U, V \subset X$ such that $U \cup V = X$. Let $a \in U$ and $b \in V$. Suppose γ is a path from a to b Then the image of γ is connected because I is. But $U \cap \gamma(I)$ and $V \cap \gamma(I)$ are disjoint and relatively open in $\gamma(I)$, so $\gamma(I)$ is separated, producing a contradiction. □

Example 9. Let $X = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1] \cap \mathbb{Q} \text{ and } y \in [0, 1]\}$. Then X is connected but not path connected.

5. Compactness

5.1. Covers.

Definition 31. Let X be a space and $A \subset X$. A *cover* of A is a collection of sets $\mathcal{E} \subset \mathcal{P}(X)$ such that $A \subset \cup\mathcal{E}$. If each set $E \in \mathcal{E}$ is open, then $\mathcal{E}$ is called and *open cover*. If only finitely many sets are in $\mathcal{E}$, then $\mathcal{E}$ is called a *finite cover*. Notice that the word finite applies to $\mathcal{E}$ whereas the word open applies to the sets in $\mathcal{E}$.

If $\mathcal{E}$ is a cover of A, a *subcover* of $\mathcal{E}$ is a subset of $\mathcal{E}$ which is itself a cover.

The set A is *compact* if every open cover of A contains a finite subcover.

Proposition 60. *The compact subsets of a discrete space are the finite subsets. In a trivial space or a cofinite space, every subset is compact.*

Proposition 61. *Finite subsets of any space are compact.*

Definition 32. Let X be a space and let $A \subset X$. A *basis cover* of a A is a cover by members of a fixed basis for the topology on X.

Proposition 62. *Let X be a space and $A \subset X$. Then A is compact if and only if every basis cover of A contains a finite subcover.*

Proof. The forward direction is immediate, because a basis cover is an open cover.

For the other direction, let $\mathcal{B}$ be a basis and suppose that every cover of A by members of $\mathcal{B}$ has a finite subcover. Let $\mathcal{U}$ be an open cover of A. Then each member of $U \in \mathcal{U}$ is the union of members of $\mathcal{B}$. Let $\mathcal{E}_U$ be a collection of basis sets such that $\cup\mathcal{E}_U = U$. Let $\mathcal{E}$ the be union of these collections. Then $\mathcal{E}$ is a basis cover and thus has a finite subcover $\mathcal{D}$. Each set in $\mathcal{D}$ is contained in one of the original open sets in $\mathcal{U}$; the collection of these sets is now a finite subcollection of $\mathcal{U}$ which covers A. □

Definition 33. Let X be a space, $\mathcal{E} \subset \mathcal{P}(X)$, and $Y \subset X$. The *relative collection* of $\mathcal{E}$ on Y is

$$\mathcal{E} \cap Y = \{E \cap Y \mid E \in \mathcal{E}\}.$$

Proposition 63. *If X is a space and $A \subset Y \subset X$, then A is compact in X if and only if A is compact with respect to the subspace topology on Y.*

Proof. Let $\mathcal{U} \subset \mathcal{P}(X)$ be an open cover of A. Since $A \subset Y$, $\mathcal{U} \cap Y$ is an open cover of A in Y, and all such covers are of this form. Let $\mathcal{V} \subset \mathcal{U}$ so that $(\mathcal{V} \cap Y) \subset (\mathcal{U} \cap Y)$. Then $\mathcal{V}$ covers A if and only if $\mathcal{V} \cap Y$ covers A. □

Proposition 64. *A closed subset of a compact set is compact.*

Proof. Let X be a space and let K be a compact subset of X. Let $F \subset K$ be closed. Let $\mathcal{U}$ be an open cover of F. Then $\mathcal{U} \cup \{X \smallsetminus F\}$ is an open cover for K and thus has a finite subcover. If $X \smallsetminus F$ is in this subcover of K, remove it; now we have a finite subcover of F. □

Proposition 65. *The continuous image of a compact set is compact.*

Proof. Let $f : X \to Y$ be a continuous function and let $K \subset X$ be compact. Let $\mathcal{V}$ be an open cover of $f(K)$. Let $\mathcal{U} = \{f^{-1}(V) \mid V \in \mathcal{V}\}$. Then $\mathcal{U}$ is an open cover of K, and has a finite subcover $\mathcal{M}$. Suppose $|\mathcal{U}| = n$, and enumerate $\mathcal{U}$ so that $\mathcal{U} = \{U_i \mid i = 1, \ldots, n\}$. For each i, select $V_i \in \mathcal{V}$ such that $U_i = f^{-1}(V_i)$. Let $\mathcal{W} = \{V_i \mid i = 1, \ldots, n\}$. Then $\mathcal{W} \subset \mathcal{V}$ is a finite subcover of $f(K)$. □

Proposition 66. *Let $\mathcal{X}$ be a family of spaces and let $\times\mathcal{X}$ be the Cartesian product endowed with the product topology. If every $X \in \mathcal{X}$ is compact, then $\times\mathcal{X}$ is compact.*

5.2. **Finite Intersection Property.**

Definition 34. Let $\mathcal{C}$ be a collection of sets. We say that $\mathcal{C}$ has the *finite intersection property* if for every $\mathcal{D} \subset \mathcal{C}$,

$$|\mathcal{D}| < \infty \Rightarrow \cap\mathcal{D} \neq \varnothing.$$

Proposition 67. *A space X is compact if and only if every collection of closed subsets of X with the finite intersection property has nonempty intersection.*

Proof.

($\Rightarrow$) Let $\mathcal{F}$ be a collection of closed sets in X with the finite intersection property but empty intersection. Then $\mathcal{U} = \{X \smallsetminus F \mid F \in \mathcal{F}\}$ is an open cover for X. Let $\mathcal{V} \subset \mathcal{U}$ be a finite subcollection. Then $\mathcal{C} = \{X \smallsetminus V \mid V \in \mathcal{V}\}$ is a finite subcollection of $\mathcal{F}$ and so has nonempty intersection. Since $\cup\mathcal{V} = X \smallsetminus \cap\mathcal{C}$, $\mathcal{V}$ is not a cover for X so X is not compact.

($\Leftarrow$) Suppose that X is not compact and let $\mathcal{U}$ be an open cover with no finite subcover. Let $\mathcal{F} = \{X \smallsetminus U \mid U \in \mathcal{U}\}$. Let $\mathcal{C} \subset \mathcal{F}$ be a finite subcollection and let $\mathcal{V} = \{X \smallsetminus C \mid C \in \mathcal{C}\}$. Then $\cap\mathcal{C} = X \smallsetminus \cup\mathcal{V}$ and since $\mathcal{V}$ does not cover X, $\cap\mathcal{C}$ is nonempty. Thus $\mathcal{F}$ has the finite intersection property, but since $\mathcal{U}$ covers X, $\cap\mathcal{F} = \varnothing$. □

5.3. **Sequential Compactness.**

Definition 35. A space is called *sequentially compact* if every sequence of points in X has a cluster point.

Proposition 68. *A space is sequentially compact if and only if every sequence of distinct points has a cluster point.*

Proposition 69. *A space is sequentially compact if and only if every infinite subset has an accumulation point.*

Remark 13. There exist compact spaces which are not sequentially compact, and there exist sequentially compact spaces which are not compact. However, in a metric space, the notions are equivalent.

6. Separation

6.1. Separation Axioms.

Definition 36. The following conditions on a topological space X are known as *separation axioms*:
T_0: Given two points in X, at least one of them lies in an open set not containing the other.
T_1: Given two points in X, each of them lies in an open set not containing the other.
T_2: Given two points in X, there exist disjoint open sets, each containing exactly one of the points.
T_3: Given a point and a closed set in X not containing the point, there exist disjoint open sets, one containing the point and one containing the closed set.
T_4: Given two disjoint closed sets in X, there exist disjoint open sets, each containing exactly one of the closed sets.
T_5: Given two subsets $A, B \subset X$ with $(A \cap \overline{B}) \cup (\overline{A} \cap B) = \varnothing$, there exist disjoint open sets, each containing exactly one of the subsets.

We say that X is a T_n space if X satisfies the T_n axiom for $n = 0, 1, 2, 3, 4, 5$.

Proposition 70. *The separation axioms are hierarchical in the following sense:*

$$T_5 + T_1 \Rightarrow T_4 + T_1 \Rightarrow T_3 + T_1 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0.$$

Proposition 71. *The product of a family of T_n spaces is a T_n space for $n = 0, 1, 2, 3$.*

Proposition 72. *A nonempty trivial space is not a T_0 space but is T_3 and T_4.*

Proposition 73. *The natural numbers with the right order topology are a T_0 space but not a T_1 space.*

Proposition 74. *The Sierpinski space is a T_0 space which is not a T_1 space.*

Proposition 75. *An infinite cofinite space is a T_1 space but not a T_2, T_3, nor a T_4 space.*

Proposition 76. *Discrete spaces are T_5 spaces.*

6.2. Closed Point Spaces.

Definition 37. A T_1 space is called a *closed point space.*

Proposition 77. *A space X is a T_0 space if and only if $\overline{\{x\}} = \overline{\{y\}} \Rightarrow x = y$ for all $x, y \in X$.*

Proposition 78. *A space X is a T_1 space if and only if all of the singleton subsets of X are closed.*

Proposition 79. *A space X is a T_1 space if and only if all of the finite subsets of X are closed.*

Proposition 80. *A space X is a T_1 space if and only if singleton sets are equal to the intersection of all neighborhoods containing them.*

6.3. **Hausdorff Spaces.**

Definition 38. A T_2 space is called a *Hausdorff space.*

Proposition 81. *A space X is a Hausdorff space if and only if singleton sets are equal to the intersection of all closed neighborhoods containing them.*

Proposition 82. *A space X is a Hausdorff space if and only if for every pair of disjoint compact sets K_1, K_2 there exist disjoint open sets U_1, U_2 such that $K_1 \subset U_1$ and $K_2 \subset U_2$.*

Proposition 83. *A compact subset of a Hausdorff space is closed.*

Proof. Let X be a Hausdorff space and K a compact subset of X. Let $y \in X \setminus K$ and for each $x \in K$, let U_x and V_x be disjoint open sets such that $x \in U_x$ and $y \in V_x$. Then sets $\{U_x \mid x \in K\}$ cover K and thus have a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$. Then $V_y = \cap_{i=1}^n U_{x_i}$ is an open neighborhood of y which is disjoint from K. The complement of K in X is the union of all such sets, and is therefore open. Thus K is closed. □

Proposition 84. *If X is a Hausdorff space then every sequence in X which has a limit point converges.*

Proof. Let X be a Hausdorff and let $x : \mathbb{N} \to X$ be a sequence in X which has a limit point p. Let $q \in X$. If p and q are distinct, there exist disjoint neighborhoods of p and q, and x is eventually in the neighborhood of p, and so is not eventually in the neighborhood of q. Therefore q is not a limit point. □

Proposition 85. *Let X be a compact space and Y a Hausdorff space. Let $f : X \to Y$ be continuous and bijective. Then f is a homeomorphism.*

Proof. It suffices to show that f is an closed map. Since X is compact, every closed subset of X is compact. Its image in Y is compact because f is continuous. Since Y is Hausdorff, this image is closed. □

Proposition 86. *If X is a space, Y is a Hausdorff space, and $f : X \to Y$ is continuous, then the set $D = \{(x, f(x)) \mid x \in X\}$ is a closed subset of $X \times Y$.*

Proof. Let $Z = X \times Y$ and let $(x, y) \in Z \setminus D$. Since Y is Hausdorff, there exist disjoint open neighborhoods U of $f(x)$ and V of y. Since $f^{-1}(U)$ contains x and is disjoint from $f^{-1}(V)$, x is not a limit point of $f^{-1}(V)$ and so $x \notin \overline{f^{-1}(V)}$. Let $O = f^{-1}(U) \setminus \overline{f^{-1}(V)}$. Then O is an open set containing x so $(x, y) \in O \times V$. Also $O \times V$ is disjoint from D.

This shows that very point is the complement of D is contained in an open set disjoint from D. Therefore D is closed. □

Corollary 4. *A space X is Hausdorff if and only if the diagonal of $X \times X$ is closed.*

Proof. Suppose X is Hausdorff. Then the identity map on X is continuous, so $D = \{(x, \mathrm{id}(x)) \mid x \in X\} = \{(x, x) \mid x \in X\}$ is closed.

Let X be a space and suppose that $D = \{(x, x) \mid x \in X\}$ is closed. Then for $y \neq x$ there exists of basis neighborhood $U \times V$ of (x, y) such that $U \times V \cap D = \varnothing$, where U and V are open subsets of X with $x \in U$ and $y \in V$.

Since $(x, x) \notin U \times V$, $x \notin V$. Since $(y, y) \notin U \times V$, $y \notin U$. Thus X is Hausdorff. □

6.3.1. *The Tube Lemma.* We prove the following:

Proposition 87. *Let X and Y be topological spaces, with Y compact and Hausdorff. Let $f : X \to Y$. Then f is continuous if and only if the graph of f is closed.*

Definition 39. Let X and Y be topological spaces and let $x_0 \in X$. The *fiber* over x in $X \times Y$ is the set $\{x\} \times Y$. A *tube* over x in $X \times Y$ is a set of the form $A \times Y$, where $x \in A \subset X$.

Lemma 1 (Tube Lemma)**.** *Let X and Y be topological spaces, with Y compact. Let $x \in X$ and let G be an open subset which contains the fiber over x. Then G contains an open tube over x.*

Proof. Sets of the form $U \times V$, where U is open in X and V is open in Y, form a basis for the topology of $X \times Y$. Thus G is the union of such sets.

For each $y \in Y$, select a basis neighborhood $U_y \times V_y \subset G$ of (x, y) in the fiber over x. The V_y form a cover of Y, and thus have a finite subcover $V_1, \ldots, V_n$; let $U_1, \ldots, U_n$ be the corresponding subsets of X so that $U_i \times V_i$ cover the fiber over x. Let $U = \cap_{i=1}^n U_i$; then U is open in X and contains x. Now $\cup_{i=1}^n U \times V_i = U \times Y \subset G$ is a tube over x which is contained in G. □

Lemma 2. *Let X and Y be topological spaces, with Y compact. Then the projection $\pi_Y : X \times Y \to Y$ given by $(x, y) \mapsto y$ is a closed map.*

Proof. Let $F \subset X \times Y$ be a closed set in $X \times Y$, and let $x \in X \setminus f(F)$. Since the complement of F in $X \times Y$ is an open set which contains the fiber over x, it contains a tube over x. The projection of this tube onto X is an open set containing x which is disjoint from $f(F)$. Thus the complement of $f(F)$ in X is open. □

Proposition 88. *Let X and Y be topological spaces, with Y compact. Let $f : X \to Y$ be a function whose graph is closed. Then f is continuous.*

Proof. It suffices to show that the inverse image of a closed set is closed. Let C be a closed set in Y. Since projection is continuous, $\pi_Y^{-1}(C)$ is closed in $X \times Y$. Let F be the intersection of this inverse image with the graph of f. Then F is closed in $X \times Y$. Since Y is compact, the projection π_X is a closed map, so $\pi_X(F)$ is closed in X. But $\pi_X(F) = f^{-1}(C)$. □

Lemma 3. *Let X and Y be topological spaces. Let $f : X \to Y$ be a continuous function. Let $g : X \to X \times Y$ be the function given by $x \mapsto (x, f(x))$. Then g is continuous.*

Proof. Let G be an open set in $X \times Y$. Intersection G with the graph of f if $g^{-1}(G)$ is empty, then it is open, so assume that it is nonempty and let $x \in g^{-1}(G)$. Let $U \times V$ be a basis neighborhood of $(x, f(x))$ which is contained in G. Then $f^{-1}(V) \cap \pi_X(G)$ is open in X, $x \in f^{-1}(V) \cap \pi_X(G)$, and $f^{-1}(V) \cap \pi_X(G)$ is contained in $g^{-1}(G)$. □

Proposition 89. *Let X and Y be topological spaces, with Y Hausdorff. Let $f : X \to Y$ be a continuous function. Then the graph of f is closed.*

Proof. Let F be the graph of f. Let $(x, y) \in X \times Y \setminus F$. Then $y \neq f(x)$; let V_1 and V_2 be open neighborhoods in Y of y and $f(x)$ respectively which are disjoint. Let $g : X \to X \times Y$ be the map given by $x \mapsto (x, f(x))$. Then $U = g^{-1}(X \times V_2)$ is open in X, and $U \times V_1$ is an open neighborhood of (x, y) which does not intersect F. Thus the complement of F is open. □

6.4. **Regular Spaces.**

Definition 40. A T_3 space which is also a T_1 space is called *regular*.

Proposition 90. *A space X is regular if and only if every neighborhood of a point in X contains a closed neighborhood.*

Proposition 91. *A space X is a regular space if and only if for every pair of disjoint sets K, F such that K is compact and F is closed there exist disjoint open sets U, V such that $K \subset U$ and $F \subset V$.*

Proposition 92. *A compact Hausdorff space is regular.*

6.5. **Normal Spaces.**

Definition 41. A T_4 space which is also a T_1 space is called *normal*.

Proposition 93. *A space X is normal if and only if for every closed set F in X and every open set U in X containing F there exists an open set V such that $F \subset V \subset \overline{V} \subset U$.*

Proposition 94. *A compact regular space is normal.*

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